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CERTAIN SOLUTIONS OF THE EQUATIONS OF LAMINAR BOUNDARY LAYER WITH LARGE BLOWING

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V. N. FILIMONOV
(Sevastopol')

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The laminar boundary layer is studied for a binary mixture in the case when large blowing takes place from the streamlined surface. Velocity, concentration and temperature distributions within the boundary layer are obtained, formulas for computing the distance to the "line of spreading" are given and expressions for the velocity, concentration and temperature gradients at the surface of the body related to the magnitude of the blowing, are derived.

It was shown earlier [1] that the concentration and temperature gradients at the separation point on the body decrease exponentially with increasing blowing; the author of [2] obtained the power dependence on the blowing everywhere, except at the separation point. The present paper gives expressions containing both these results and an estimate of the region of validity for each of them.

1. The laminar boundary layer equations for a binary mixture have the form [3]

$$(l f''_{\eta\eta})_\eta' + f f''_{\eta\eta} + \Lambda\left[\frac{\rho_e}{\rho} - (f_\eta')^2\right] = 2\xi\,(f_\eta' f''_{\xi\eta} - f_\xi' f''_{\eta\eta})$$

$$\left(\frac{l}{S}\,c_\eta'\right)_\eta' + f c_\eta' = 2\xi\,(f_\eta' c_\xi' - f_\xi' c_\eta') \qquad (1.1)$$

$$\left(\frac{l c_p}{\sigma}\,\theta_\eta'\right)_\eta' + c_p f \theta_\eta' + \frac{l}{S}(c_{p1} - c_{p2})\,c_\eta' \theta_\eta' + l\,\frac{u_e^2}{T_e}\,(f''_{\eta\eta})^2 =$$

$$= f_\eta'\left(\beta c_p \theta + \Lambda\,\frac{\rho_e}{\rho}\,\frac{u_e^2}{T_e}\right) + 2\xi c_p\,(f_\eta' \theta_\xi' - f_\xi' \theta_\eta')$$

$$\eta = \frac{r^k u_e}{\sqrt{2\xi}}\int_0^y \rho\,dy, \qquad \xi = \int_0^x \rho_e \mu_e u_e r^{2k}\,dx$$

$$\Lambda(\xi)=\frac{2\xi}{u_e}\frac{du_e}{d\xi},\qquad \beta(\xi)=\frac{2\xi}{T_e}\frac{dT_e}{d\xi},\qquad l=\frac{\mu\rho}{\mu_w\rho_w}$$

Here η and ξ are dimensionless coordinates along the normal and the surface, respectively, f is the modified stream function ($f_\eta' = u\ /\ u_e$ is the dimensionless velocity), $\theta = T\ /\ T_e$ is the dimensionless temperature, c and c_{p_1} denote the concentration and the heat capacity of a single component, c_p and ρ denote the heat capacity and the density of the mixture, S is the Schmidt number, σ is the Prandtl number, $k = 0$ and $k = 1$ for the plane and the axisymmetric flow, respectively. The subscript e denotes the outer boundary of the boundary layer and w denotes the surface of the body.

The boundary conditions at the surface of the body are

$$f_\eta' = 0,\qquad c = c_w(\xi),\qquad \theta = \theta_w(\xi)\qquad \text{when } \eta = 0$$

$$f(\xi, 0) + 2f^{\xi\prime}(\xi, 0) = G(\xi) = -\frac{\sqrt{2\xi}}{\xi_x'} r^k (\rho v)_w$$

and at the outer boundary of the boundary layer

$$f_\eta' = 1,\qquad c = c_e,\qquad \theta = 1\quad \text{when } \eta = \infty$$

Here $G(\xi)$ is a known smooth function. The condition of large blowing means that the absolute magnitude of $G(\xi)$ is so large, that $1/G(0)$ can be regarded as a small parameter.

To obtain an explicit expression for the small parameter in (1.1), we introduce a new variable

$$h = f(\xi, \eta)\ /\ f(\xi, 0) \tag{1.2}$$

and new unknown functions

$$z(\xi, h) = \left(\frac{u}{u_e}\right)^2 = f^2(\xi, 0)(h_\eta')^2,\quad c(\xi, h) = c(\xi, \eta(\xi, h)),\quad \theta(\xi, h) = \theta(\xi, \eta(\xi, h)) \tag{1.3}$$

Then the system (1.1) becomes

$$A(\xi)\,hz_h' - 2\xi z_\xi' - 2\Lambda\left(z - \frac{\rho_e}{\rho}\right) = -\frac{\sqrt{z}}{\alpha^2 a^2(\xi)}(lz_h')_h'$$

$$A(\xi)\,hc_h' - 2\xi c_\xi' = -\frac{1}{\alpha^2 a^2(\xi)}\left(\frac{l}{S}\sqrt{z}c_h'\right)_h' \tag{1.4}$$

$$A(\xi)\,h\theta_h' - 2\xi\theta_\xi' - \frac{\Lambda}{c_p}\frac{\rho_e}{\rho}\frac{u_e^2}{T_e} - \beta\theta = -\frac{1}{\alpha^2 a^2(\xi)\,c_p}\times$$

$$\times\left[\left(\frac{lc_p}{\sigma}\sqrt{z}\theta_h'\right)_h' + \frac{l}{S}(c_{p1} - c_{p2})\sqrt{z}c_h'\theta_h' + \frac{l}{4\sqrt{z}}\frac{u_e^2}{T_e}(z_h')^2\right]$$

$$\alpha = -f(0, 0) = -G(0),\qquad a(\xi) = \frac{f(\xi, 0)}{f(0, 0)},\qquad A(\xi) = 1 + \frac{2\xi}{a(\xi)}\frac{da}{d\xi}$$

with the boundary conditions

$$z(\xi, 1) = 0,\qquad c(\xi, 1) = c_w(\xi),\qquad \theta(\xi, 1) = \theta_w(\xi)$$

$$z(\xi, -\infty) = 1,\qquad c(\xi, -\infty) = c_e,\qquad \theta(\xi, -\infty) = 1$$

The quantity $1/\alpha^2$ is the small parameter. The problem formulated pertains to the class of problems of equations with a small parameter accompanying the higher order derivatives, the solution of which suffers a discontinuity, in the limit, at an internal point of the region [4]. An asymptotic solution of this problem is obtained below in Sect. 2.

2. The equation of state makes it possible to express the quantity $\rho_e\ /\ \rho$ for a binary mixture in the form of a product

$$\frac{\rho_e}{\rho} = \theta\lambda(c), \qquad \lambda(c) = \frac{1 + (\gamma - 1)c}{1 + (\gamma - 1)c_e} \tag{2.1}$$

Here $1 / \gamma$ is the ratio of molecular weights of the two components, their respective concentrations equal to c and $(1 - c)$. Inserting (2.1) into (1.4) we solve the degenerate (with $\alpha^{-2} = 0$) problem, first for the concentration, then for the temperature and finally for the velocity. We precede this bv introducing the function F in the following manner: $F(t) = \xi$ if $t = \xi a^2(\xi)$, i.e. $F(t)$ is a solution of the equation $Fa^2(F) = t$. Taking into account the fact that $a(\xi) = f(\xi, 0) / f(0, 0)$ can easily be obtained from the boundary condition

$$f(\xi, 0) + 2\xi f_\xi'(\xi, 0) = G(\xi)$$

we shall regard the function F as known for each concrete problem. Thus, e.g. for the constant rate of blowing $(f(\xi, 0) = f(0, 0))$ we obtain $F(t) \equiv t$.

When $\alpha^{-2} = 0$, the solution of (1.4) with (2.1) taken into account has the form

$$c = (1 - U)c_w(F_*) + c_e U$$

$$\theta = U + (1 - U)\frac{T_e(F_*)}{T_e(\xi)}\theta_w(F_*)\exp\left[\frac{\nu}{2c_p}\int_\xi^{F_*}(u_e^2)_t' T_e'(t)\,dt\right]$$

$$z = U + (1 - U)2c_p\frac{T_e(\xi)}{u_e^2(\xi)}\left[\frac{T_e(F_*)}{T_e(\xi)}\theta_w(F_*) - \theta(\xi, h)\right] \tag{2.2}$$

where $F_* = F(\xi a^2(\xi)h^2)$ and $U = U(h)$ is a symmetrical unit function

$$U(h) = \begin{cases} 0, & h < 0 \\ 1/2, & h = 0 \\ 1, & h > 0 \end{cases}$$

We assume here that c_p depends only on concentration. For an incompressible medium $(\rho_e / \rho = 1)$ the degenerate equation for the velocity becomes independent of the other two equations. The solution has the form

$$z = 1 - (1 - U)\frac{u_e^2(F_*)}{u_e^2(\xi)}$$

The absence of a discontinuity on the line $h = 0$ represents an interesting feature of this solution. According to [4], the formulas (2.2) yield the limiting form of the actual solution ontside the line $h = 0$, when $\alpha \to \infty$. A more accurate approximation to the actual solution can be obtained by seeking the solutions of (1.4) in the form of asymptotic series

$$(z, c, \theta) = (z_0, c_0, \theta_0) + \frac{1}{\alpha^2}(z_1, c_1, \theta_1) + \ldots \tag{2.3}$$

It is evident that z_0, c_0 and θ_0 are given by (2.2). For c_1, θ_1 and z_1 the solution near the surface of the body is as follows:

$$c_1 = \frac{1}{2}\int_{F_*}^{\xi} A(t, h_*)\,t^{-1}a^{-2}(t)\,dt$$

$$\theta_1 = \frac{1}{2T_e(\xi)}e^{-\omega(\xi)}\int_{F_*}^{\xi} B(t, h_*)\,e^{\omega(t)}T_e(t)\,t^{-1}a^{-2}(t)\,dt \tag{2.4}$$

$$z_1 = \frac{1}{2u_e^2(\xi)}\int_{F_*}^{\xi} C(t, h_*)\,u_e^2(t)\,t^{-1}a^{-2}(t)\,dt$$

$$A = \left(\frac{l}{S}\sqrt{z}c_h'\right)_h', \qquad C = \sqrt{z}\,(lz_h')_h' + 2a^2\Lambda\,(\theta_0\lambda_1 + \theta_1\lambda_0)$$

$$B = \frac{1}{c_p^{(0)}}\left[\left(\frac{lc_p}{\sigma}\sqrt{z}\theta_h'\right)_h' + \frac{l}{S}(c_{p1} - c_{p2})\sqrt{z}c_h'\theta_h' + \frac{l}{4\sqrt{z}}\frac{u_e^2}{T_e}(z_h')^2\right] -$$

$$- a^2\frac{\Lambda}{c_p^{(0)}}\frac{u_e^2}{T_e}\left[\theta_0\lambda_1 - \theta_0\lambda_0\frac{c_p^{(1)}}{c_p^{(0)}}\right]$$

$$\omega(\xi) = \frac{\lambda_0}{2c_p^{(0)}}\int_0^\xi (u_e^2)_t' T_e^{-1}(t)\,dt, \qquad h_* = h\left(\frac{\xi a^2(\xi)}{ta^2(t)}\right)^{1/2}$$

where the following expansions are utilized

$$c_p(c) = c_{p1}c + c_{p2}(1 - c) = [c_{p2} + (c_{p1} - c_{p2})c_0] + \alpha^{-2}(c_{p1} - c_{p2})c_1 + \ldots =$$

$$= c_p^{(0)} + \alpha^{-2}c_p^{(1)} + \ldots, \qquad \rho_e/\rho = \theta_0\lambda_0 + \alpha^{-2}(\theta_1\lambda_0 + \theta_0\lambda_1) + \ldots$$

$$\lambda_0 = \frac{1 + (\gamma - 1)c_0}{1 + (\gamma - 1)c_e}, \qquad \lambda_1 = \frac{(\gamma - 1)c_1}{1 + (\gamma - 1)c_e}$$

$$(l, S, \sigma) = (l(c_0, \theta_0), S(c_0, \theta_0), \sigma(c_0, \theta_0)) + O(\alpha^{-2})$$

The function A is determined by c_0, θ_0 and z_0, the function B by c_0 θ_0, z_0 and c_1 and the function C by c_0, θ_0, z_0, c_1 and θ_1. Near the outer boundary for $i \geqslant 1$ we have $c_i = \theta_i = z_i = 0$.

The solutions (2.2) and (2.4) with (1.2) and (1.3) taken into account, make it possible to obtain the velocity, concentration and temperature derivatives with respect to η Thus at the surface of the body we have

$$u_\eta'(\xi, 0) = -G^{-1}(\xi)\,\Lambda\frac{\rho_e}{\rho_w} + O(\alpha^{-5})$$

$$c_\eta'(\xi, 0) = G^{-3}(\xi)\frac{\Lambda}{S_w}\frac{\rho_e}{\rho_w}2\xi c_{w\xi}' + O(\alpha^{-5}) \qquad (2.5)$$

$$\theta_\eta'(\xi, 0) = G^{-3}(\xi)\left[\frac{\Lambda}{\sigma_w}\frac{\rho_e}{\rho_w}(2\xi\theta_{w\xi}' + \beta\theta_w) + \left(\frac{1}{\sigma_w} - 1\right)\frac{u_e^2}{T_e}\frac{\Lambda}{c_{pw}}\left(\frac{\rho_e}{\rho_w}\right)^2\right] + O(\alpha^{-5})$$

Similar results at the surface of the body were obtained in [2] directly from the initial equations. Comparison with numerical results [2] shows good agreement even for $\alpha \geqslant 2$. We can use this fact to assert that the solutions (2.2), (2.4) can be used in the case of completely real blowing.

The last expression of (2.2) yields the distance η_0 to the line of spreading $h = 0$

$$\eta_{(0)}(\xi) = -f(\xi, 0)\int_0^1 \frac{dh}{\sqrt{z(h)}} \qquad (2.6)$$

At the point $\xi = 0$ we have

$$\eta_{(0)}(0) = \frac{\alpha}{2\Lambda}B\left(\frac{1}{2\Lambda}, \frac{1}{2}\right)\left(\frac{\rho e}{\rho_w}\right)^{-1/2} + \left(\frac{\rho e}{\rho_w}\right)^{-1/2}\left\{\left[1 + O\left(\frac{2\Lambda - 1}{\alpha^{2\Lambda - 1}}\right)\right]^{-1/2} - 1\right\} + O(\alpha^{-2\Lambda}) \qquad (2.7)$$

where B is the beta function.

In Sect. 3 which follows, we seek a solution yielding more accurate quantities $\eta_{(0)}$, $(c_\eta')_w$ and $(\theta_\eta')_w$.

3. As we know [1], in the vicinity of the front stagnation point ($\xi = 0$) the quantities $(c_\eta')_w$ and $(\theta_\eta')_w$ are exponentially small in α^{-2}. Since the series solution (2.3) does not yield such a relationship, we propose to seek a solution of the second and third equation of (1.4) in the form of a sum of two functions, $c = c_{(1)} + c_{(2)}$ and $\theta = \theta_{(1)} + \theta_{(2)}$ such, that $c_{(1)}$ and $\theta_{(1)}$ satisfy the equations

$$A(\xi)\,hc'_{(1)h} = -\frac{1}{\alpha^2 a^2(\xi)}\left(\frac{l}{S}\sqrt{z}\,c'_{(1)h}\right)'_h \tag{3.1}$$

$$A(\xi)\,h\theta'_{(1)h} = -\frac{1}{\alpha^2 a^2(\xi)\,c_p}\left[\left(\frac{lc_p}{\sigma}\sqrt{z}\,\theta'_{(1)h}\right)'_h + \frac{l}{S}(c_{p1}-c_{p2})\sqrt{z}\,c_h{}'\theta'_{(1)h}\right]$$

with the boundary conditions

$$c_{(1)}(\xi,1) = c_w(0), \qquad \theta_{(1)}(\xi,1) = \theta_w(0)$$
$$c_{(1)}(\xi,-\infty) = c_e, \qquad \theta_{(1)}(\xi,-\infty) = 1$$

Then $c_{(2)}$ and $\theta_{(2)}$ will satisfy the equations

$$A(\xi)\,hc_{(2)h} - 2\xi c'_{(2)\xi} = 2\xi c'_{(1)\xi} - \frac{1}{\alpha^2 a^2(\xi)}\left(\frac{l}{S}\sqrt{z}\,c'_{(2)h}\right)'_h$$

$$A(\xi)\,h\theta'_{(2)h} - 2\xi\theta'_{(2)\xi} - \beta\theta_{(2)} = \beta\theta_{(1)} + 2\xi\theta'_{(1)\xi} + \frac{\Lambda}{c_p}\frac{u_e^2}{T_e}\frac{\rho_e}{\rho} - \tag{3.2}$$
$$-\frac{1}{\alpha^2 a^2(\xi)\,c_p}\left[\left(\frac{lc_p}{\sigma}\sqrt{z}\,\theta'_{(2)h}\right)'_h + \frac{l}{S}(c_{p1}-c_{p2})\sqrt{z}\,c'_h\theta'_{(2)h} + \frac{l}{4\sqrt{z}}\frac{u_e^2}{T_e}(z_h{}')^2\right]$$

with the boundary conditions

$$c_{(2)}(\xi,1) = c_w(\xi) - c_w(0), \qquad \theta_{(2)}(\xi,1) = \theta_w(\xi) - \theta_w(0)$$
$$c_{(2)}(\xi,-\infty) = \theta_{(2)}(\xi,-\infty) = 0$$

The coefficients l, S etc. themselves are in fact dependent on c and θ, therefore such a separation can only be performed formally, assuming that all coeffcients in (1.4) are known functions of ξ and h. At some distance from the line $h = 0$, these coefficients can be computed using the solutions (2.2) and (2.4). The solutions $c_{(2)}$ and $\theta_{(2)}$ should be determined after $c_{(1)}$ and $\theta_{(1)}$ have been found. Then the corresponding terms in (3.2) dependent on $c_{(1)}$ and $\theta_{(1)}$ will be known functions of ξ and h. Solutions of the Eqs. (3.1) can be written in the form

$$c_{(1)} = c_w(0) + (c_e - c_w(0))\int_1^h \frac{S}{l\sqrt{z}}\exp\psi_c(\xi,h')\,dh' \Big/ \int_1^{-\infty}\frac{S}{l\sqrt{z}}\exp\psi_c(\xi,h)\,dh$$

$$\theta_{(1)} = \theta_w(0) + (1-\theta_w(0))\int_1^h \frac{\sigma}{lc_p\sqrt{z}}\exp\psi_\theta(\xi,h')\,dh' \Big/ \int_1^{-\infty}\frac{\sigma}{lc_p\sqrt{z}}\exp\psi_\theta(\xi,h)\,dh \tag{3.3}$$

$$\psi_\theta(\xi,h) = \alpha^2 a^2(\xi)\,A(\xi)\int_h^1 \frac{\sigma}{l\sqrt{z}}h'dh' + \int_h^1 \frac{c_{p1}-c_{p2}}{c_p}Lc_{h'}{}'dh'$$

$$\psi_c(\xi,h) = \alpha^2 a^2(\xi)\,A(\xi)\int_h^1 \frac{S}{l\sqrt{z}}h'dh', \qquad L = \frac{\sigma}{S}$$

from which we have

$$(c_{(1)n})_w = \left(-\frac{\sqrt{z}}{\alpha a(\xi)}c'_{(1)h}\right)_{h=1} = \frac{S}{\alpha a(\xi)}(c_e - c_w(0)) \Big/ \int_{-\infty}^1 \frac{S}{l\sqrt{z}}\exp\psi_c(\xi,h)\,dh$$

$$(\theta'_{(1)\eta})_w = \frac{\sigma_w}{\alpha a(\xi)\, c_{pw}}(1-\theta_w(0)) \Big/ \int_{-\infty}^{1} \frac{\sigma}{l c_p \sqrt{z}} \exp \psi_\theta(\xi, h)\, dh \qquad (3.4)$$

Here L is the Lewis number. Solutions (2.2) are used to compute the integrals in (3.4). Then, since a large parameter α^2 appears in the exponential curve index, the Laplace formula is employed to perform the asymptotic integration [3] which yields

$$(c'_{(1)\eta})_w = \frac{S_w}{C_1}(c_e - c_w(0))\left(\frac{G(\xi)}{f(\xi,0)}\right)^{1/2} \exp\left[-G(\xi) f(\xi,0)\int_0^1 \frac{S}{l\sqrt{z}} h dh\right]$$

$$C_1(\xi) = \sqrt{\frac{\pi}{2}}\left[\left(\frac{S}{l\sqrt{z}}\right)_{(0)}^{1/2} + \left(\frac{S}{l}\right)_e^{1/2}\right] \qquad (3.5)$$

Here the subscript (0) indicates that the quantity in question is taken at $h = 0$. The expression for $(\theta'_{(1)\eta})_w$ is obtained from (3.5) by replacing S by σ and $C_1(\xi)$ by $C_2(\xi)$

$$C_2(\xi) = \sqrt{\frac{\pi}{2}}\, c_{pw}(\xi)\left[\frac{1}{c_{p(0)}}\left(\frac{\sigma}{l\sqrt{z}}\right)_{(0)}^{1/2} + \frac{1}{c_{pe}}\left(\frac{\sigma}{l}\right)_e^{1/2} e^{\varphi_1}\right] e^{\varphi_2}$$

$$\varphi_1 = (c_{p1} - c_{p2})\int_0^1 \frac{L}{c_p}(c_w(F_*))_h' dh, \quad \varphi_2 = \frac{1}{2}(c_{p1} - c_{p2})(c_e - c_w(0)) \times$$

$$\times\left[\left(\frac{L}{c_p}\right)_{(0)} - \left(\frac{L}{c_p}\right)_e\right]$$

Expressions of the type $(S/l\sqrt{z})_{(0)}$ and the integrals from zero to unity are obtained using the solutions (2.2) and (2.4) on the interval $0 < h \leqslant 1$. At the front stagnation point $(\xi = 0)$ we have

$$C_1(0) = \sqrt{\frac{\pi}{2}}\left[S_w^{1/2}\left(\frac{\rho_e}{\rho_w}\right)^{-1/4} + S_e^{1/2} l_e^{-1/2}\right]$$

$$C_2(0) = \sqrt{\frac{\pi}{2}}\left[\sigma_w^{1/2}\left(\frac{\rho_e}{\rho_w}\right)^{-1/4} + \sigma_e^{1/2} l_e^{-1/2}\frac{c_{pw}}{c_{pe}} \exp\left\{\frac{1}{2}(c_{p1} - c_{p2})(c_e - c_w(0)) \times\right.\right.$$

$$\left.\left.\times\left[\left(\frac{L}{c_p}\right)_w - \left(\frac{L}{c_p}\right)_e\right]\right\}\right]$$

and, similarly to [1], we obtain

$$\int_0^1 \frac{S}{l\sqrt{z}} h dh = S_w(0)\int_0^1 [z_0 + \alpha^{-2} z_1 + O(\alpha^{-4})]^{-1/2} h dh = S_w(0)\left[\frac{1}{2\Lambda} B\left(\frac{1}{2}, \frac{1}{\Lambda}\right) \times\right.$$

$$\left.\times\left(\frac{\rho_e}{\rho_w}\right)^{-1/2} - \frac{\Lambda^*}{\alpha^2}\right] + O(\alpha^{-4}), \qquad \Lambda^* = \frac{\Lambda - 1/2}{2-\Lambda}\left[\psi\left(\frac{3}{2}\right) - \psi\left(1+\frac{1}{\Lambda}\right)\right]$$

where ψ is the psi function.

The small parameter ceases to exert an influence in the $(1/\alpha)$-neighborhood of the line $h = 0$ and the asymptotic solutions of Sect. 2 are no longer valid. Therefore, in order for the formulas (3.3) and (3.4) to yield quantitatively correct results, we must obtain a solution near the line $h = 0$. The case of homogeneous medium in the Appendix illustrates the method of obtaining an approximate solution near this line. Applying the same method here, we obtain the following functions for the separation point near the line $h = 0$:

$$z = \left(\frac{\rho_e}{\rho}\right)_{(0)} - \left[\left(\frac{\rho_e}{\rho}\right)_{(0)} - \frac{\rho_e}{\rho_w}\right]\Phi\left(\frac{\alpha h}{\sqrt{2}}\left(\frac{\rho_e}{\rho_w}\right)^{-1/4}\right), \qquad h \geqslant 0 \qquad (3.6)$$

$$z = \left(\frac{\rho_e}{\rho}\right)_{(0)} - \left[1 - \left(\frac{\rho_e}{\rho}\right)_{(0)}\right] \Phi\left(\frac{\alpha h}{\sqrt{2}} l_e^{-1/2}\right), \qquad h \leqslant 0$$

$$\left(\frac{\rho_e}{\rho}\right)_{(0)} = \frac{\rho_e}{\rho}\left(\frac{c_e + c_w}{2}, \frac{1+\theta_w}{2}\right)$$

$$c = \frac{c_e + c_w}{2} - \frac{c_e - c_w}{2} \Phi\left(\frac{\alpha h}{\sqrt{2}} S_w^{1/2} \left(\frac{\rho_e}{\rho_w}\right)^{-1/4}\right), \qquad h \geqslant 0$$

$$c = \frac{c_e + c_w}{2} - \frac{c_e - c_w}{2} \Phi\left(\frac{\alpha h}{\sqrt{2}} S_e^{1/2} l_e^{-1/2}\right), \qquad h \leqslant 0 \tag{3.7}$$

where Φ is the error integral. The solution for θ is obtained from (3.7) by replacing c_e by unity, c_w by θ_w and S by σ Equations (3.1) contain ξ only as a parameter. The functions $c_{(1)}$ and $\theta_{(1)}$ become the main contributors to the quantities $(c_\eta')_w$ and $(\theta_\eta')_w$ near the point $\xi = 0$, therefore use of the expressions (3.6) and (3.7) is suggested for computing the integrals in (3.3) and (3.4) near the line $h = 0$, assuming that c_w, θ_w, $S_w = S(c_w, \theta_w)$ etc. in these expressions are functions of ξ. Then the form of (3.5) will be preserved, with C_1 replaced by C_3

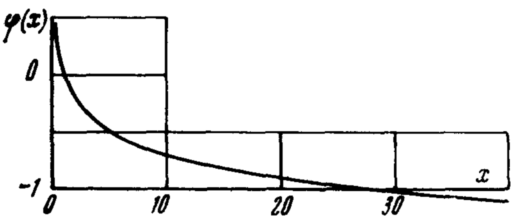

Fig. 1

$$C_3 = \int_{-\infty}^{0} \varphi_4 \exp\left[-\frac{f^2}{2}\varphi_3 + \varphi_5(f)\right] df + [\exp \varphi_5(0)] \int_0^\infty \varphi_4 \exp\left(-\int_0^f \varphi_4 df'\right) df$$

$$\varphi_3 = S_w \left(\frac{\rho_e}{\rho_w}\right)^{-1/2}, \quad \varphi_4 = \frac{S}{l\sqrt{Z}} \quad \varphi_5(f) = -\int_{-\infty}^{f} (\varphi_4 - \varphi_3) f' df'$$

and C_2 by C_4

$$C_4 = \int_{-\infty}^{0} \frac{c_{pw}}{c_p} \varphi_7 \exp\left[-\frac{f^2}{2}\varphi_6 + \varphi_9(f)\right] df + [\exp \varphi_9(0)] \int_0^\infty \frac{c_{pw}}{c_p} \varphi_7 \exp \varphi_{10} df$$

$$\varphi_6 = \sigma_w \left(\frac{\rho_e}{\rho_w}\right)^{-1/2}, \quad \varphi_7 = \frac{\sigma}{l\sqrt{Z}}, \quad \varphi_8 = \frac{c_{p1} - c_{p2}}{c_p} L c_f'$$

$$\varphi_9(f) = -\int_{-\infty}^{f} [(\varphi_7 - \varphi_6) f' + \varphi_8] df', \qquad \varphi_{10} = -\int_0^f (\varphi_7 f' + \varphi_8) df'$$

Here $Z(f) = z(-h/\alpha)$ is given by (3.6), σ, l, c_p and L are functions of $\theta(f) = \theta(-h/\alpha)$ and $c(f) = c(-h/\alpha)$ is given by (3.7). The functions (3.6) and (3.7) can also be used for the more accurate estimation of the distance to the line of spreading $h = 0$. Thus, the expressions (2.7) for the separation point should be supplemented by the following term:

$$\sqrt{2}\left(\frac{\rho_e}{\rho_w}\right)^{-1/4} \varphi\left(\left(\frac{\rho_e}{\rho}\right)_{(0)} \Big/ \frac{\rho_e}{\rho_w}\right)$$

$$\varphi(x) = \int_0^\infty \{[x + (1-x)\Phi(t)] - 1\} dt$$

Figure 1 depicts the function $\varphi(x)$.

A solution of (3.2) is sought in the form of asymptotic series in the small parameter $1 / \alpha^2$.

$$(c_{(2)}, \theta_{(2)}) = (c_{(2)0}, \theta_{(2)0}) + 1 / \alpha^2 (c_{(2)1}, \theta_{(2)1}) + \ldots$$

Near the boundary $h = 1$ we have

$$c_{(2)0} = c_w(F_*) - c_w(0) - \int_{F_*}^{\xi} \frac{\partial}{\partial t} c_{(1)}(t, h_*)\, dt$$

$$c_{(2)i}(\xi, h) = -\frac{1}{2} \int_{F_*}^{\xi} A_i(t, h_*) \frac{dt}{t} \qquad (i \geqslant 1) \tag{3.8}$$

$$\theta_{(2)0} = \frac{T_e(F_*)}{T_e(\xi)} [\theta_w(F_*) - \theta_w(0)] - \frac{1}{2T_e(\xi)} \int_{F_*}^{\xi} B_0(t, h_*)\, T_e(t) \frac{dt}{t}$$

$$\theta_{(2)i} = -\frac{1}{2T_e(\xi)} \int_{F_*}^{\xi} B_i(t, h_*)\, T_e(t) \frac{dt}{t} \qquad (i \geqslant 1)$$

and near the outer boundary of the boundary layer we have

$$c_{(2)i} = -\frac{1}{2} \int_0^{\xi} A_i(t, h_*) \frac{dt}{t}, \qquad \theta_{(2)i} = -\frac{1}{2T_e(\xi)} \int_0^{\xi} B_i(t, h_*)\, T_e(t) \frac{dt}{t}$$

Here A_i and B_i are the coefficients of expansions of the right-hand sides of (3.2) into series in $1 / \alpha^2$. Formulas (3.8) yield expressions for $(c'_{(2)\eta})_w$ and $(\theta'_{(2)\eta})_w$ at the surface of the body, which are identical with those in (2.5). Combining the results obtained we have

$$(c_\eta')_w = \frac{S_w}{C_3} (c_e - c_w) \left(\frac{G(\xi)}{f(\xi,0)}\right)^{1/2} \exp\left[-G(\xi) f(\xi, 0) \int_0^1 \frac{S}{l\sqrt{z}} h\, dh\right] +$$

$$+ G^{-3}(\xi) \frac{\Lambda}{S_w} \frac{\rho_e}{\rho_w} 2\xi c'_{w\xi} + O(\alpha^{-5})$$

$$(\theta_\eta')_w = \frac{\sigma_w}{C_4} (1 - \theta_w) \left(\frac{G(\xi)}{f(\xi, 0)}\right)^{1/2} \exp\left[-G(\xi) f(\xi, 0) \int_0^1 \frac{\sigma}{l\sqrt{z}} h\, dh\right] + \tag{3.9}$$

$$+ G^{-3}(\xi) \frac{\Lambda}{\sigma_w T_e(\xi)} \frac{\rho_e}{\rho_w} \left[2\xi T'_{w\xi} + (1 - \sigma_w) u_e^2 \frac{\Lambda}{c_{pw}} \frac{\rho_e}{\rho_w}\right] + O(\alpha^{-5})$$

These formulas show that, with increasing blowing, the concentration and temperature gradients decrease exponentially at the surface of the body near the separation point, and according to a power law further away along the surface. The latter is true if $c_w(\xi) \neq \text{const}$, $T_w(\xi) \neq \text{const}$ and $\sigma_w \neq 1$, otherwise the exponential dependence is retained over the whole surface.

Assuming the blowing fixed, let us estimate the quantity $\xi^*(\alpha)$ where both terms (power and exponential) contribute equally. Assuming that all functions defined at the surface of the body are sufficiently smooth and that ξ^* itself is small, we have

$$\xi^* \approx \left|\frac{c_e - c_w}{c_{w\xi}'}\right| \frac{\rho_w}{\rho_e} \frac{S_w^2}{2\Lambda} \frac{\alpha^3}{C_3} \exp\left[-\alpha^2 \frac{S_w}{2\Lambda} \left(\frac{\rho_e}{\rho_w}\right)^{-1/2} B\left(\frac{1}{2}, \frac{1}{\Lambda}\right) + S_w \Lambda^*\right]$$

for the concentration gradient and

$$\xi^* \approx \left|(1-\theta_w)\left[T_{w\xi}' + (1-\sigma_w)\frac{\rho_e}{\rho_w}\frac{u_e'(0)}{(c_p\mu\rho)_w}\right]^{-1}\right|\frac{\rho_w}{\rho_e}\frac{\sigma_w}{2C_1}T_e\alpha^2 \times$$
$$\times \exp\left[-\alpha^2\sigma_w\left(\frac{\rho_e}{\rho_w}\right)^{-1/2} + \sigma_w\Lambda^*\right]$$

for the temperature gradient. The last formula refers to a plane flow around a blunt body when the function $u_e^2(\xi)$ is proportional to ξ for its small value. In the more general case we have

$$\xi^* = [\alpha^{n_1}\exp(-\alpha^2 n_2)]\,/\,|O(1-\sigma_w) + O(T_{w\xi}')| \quad n_1 > 0 \quad n_2 > 0,$$

and the formula cannot be written in its exact form unless the actual form of $u_e(x)$ is known.

The last three formulas show that the region in which the exponential term in (3.9) exerts a predominant influence, decreases with increasing blowing. On the other hand, the same region increases with decreasing $c_{w\xi}'$, $T_{w\xi}'$ and $(1-\sigma_w)$.

4. Appendix. To avoid unnecessary tedium, we illustrate the method of obtaining an approximate solution near the line $h = 0$ by considering the particular case of a homogeneous medium. Then $\gamma^* = 1$ and according to (2.1) $\rho_e/\rho = \theta$. Equations (1.4) (with $l = \sigma = 1$) yield the following set of equations at the separation point $\xi = 0$:

$$hz_h' + 2\Lambda(\theta - z) = -\alpha^{-2}\sqrt{z}\,z_{hh}''$$
$$h\theta_h' = -\alpha^{-2}(\sqrt{z}\,\theta_h')_h'$$

with the boundary conditions

$$z = 0, \quad \theta = \theta_w \text{ when } h = 1$$
$$z = 1, \quad \theta = 1 \text{ when } h = -\infty$$

Such a problem was solved earlier [6] on a digital computer and it will be expedient to compare the approximate solutions obtained in the present paper with the numerical ones.

Near the boundary $h = 1$ the solution is sought in the form of asymptotic series (2.3) where

$$z_0 = \theta_w(1 - h^{2\Lambda}), \quad z_1 = \theta_w^{3/2}2\Lambda(2\Lambda - 1)h^{2\Lambda}\int_h^1(1 - t^{2\Lambda})t^{-3}\,dt$$
$$\theta_0 = \theta_w, \quad \theta_i = 0 \quad (i \geqslant 1)$$

To obtain a solution near the line $h = 0$, we extend the variable $f = -\alpha h$ and seek a solution for

$$\sqrt{Z}\,Z_{ff}'' + fZ_f' + 2\Lambda(\Theta - Z) = 0 \tag{4.1}$$
$$(\sqrt{Z}\,\Theta_f')_f' + f\Theta_f' = 0, \quad z = Z(f), \quad \theta = \Theta(f)$$

in the form of series

$$(Z, \Theta) = (Z_0, \Theta_0) + \frac{1}{\alpha^{2\Lambda}}(Z_1, \Theta_1) + \ldots \tag{4.2}$$

with the boundary conditions

$$Z_0 = \Theta_0 \to \theta_w, \quad Z_1 \to -\theta_w(-f)^{2\Lambda}, \quad \Theta_1 \to 0 \quad \text{as} \quad f \to -\infty \tag{4.3}$$
$$Z_0 = \Theta_0 \to 1, \quad Z_1 = \Theta_1 \to 0 \quad \text{as} \quad f \to \infty$$

In [6] the problem (4.1) – (4.3) is solved numerically. Below we give a method of obtaining an approximate analytic expression for the functions Z_0 and Θ_0.

We note that if θ_0 in the equation for z_0 is assumed to be a known function of h, then the function

$$z_0 = 2\Lambda h^{2\Lambda} \int_h^{-\infty} \theta_0(x)\, x^{-2\Lambda-1}\, dx$$

is a solution satisfying the condition $z_0(-\infty) = 1$ while the function

$$z_0 = 2\Lambda h^{2\Lambda} \int_h^{1} \theta_0(x)\, x^{-2\Lambda-1}\, dx$$

is a solution satisfying the condition $z_0(1) = 0$. When $h \to 0$, both these functions tend to the common quantity $\theta_0(0)$. We therefore assume that $Z_0(0) = \Theta_0(0)$ (analogous assumption that $Z_0(0) = (\rho_e / \rho)_{(0)}$ is used in deriving (3.6)). We also assume that $\Theta_0(0) = {}^1/_2\, [\Theta(-\infty) + \Theta(\infty)]$. The latter assumption is not necessary, but when used, it yields a solution which is closer to the exact one. In deriving (3.7) it is similarly assumed that $\theta_{(0)} = {}^1/_2\,(1 + \theta_w)$ and $c_0 = {}^1/_2\,(c_e + c_w)$. Furthermore, the functions Z_0 and Θ_0 in Eqs. (4.1) which are regarded as coefficients of these equations, are replaced in the region $-\infty < f < 0$ by their limiting value θ_w, and in the region $0 < f < \infty$ by unity. The resulting "linearized" equations are then solved with the boundary conditions (4.3) and a supplementary condition $Z_0(0) = \Theta_0(0) = {}^1/_2\,(1 + \theta_w)$. Then we have

$$Z_0 = \Theta_0 = \frac{1+\theta_w}{2} + \frac{1-\theta_w}{2} \times \begin{cases} \Phi(\theta_w^{-1/4} f / \sqrt{2}), & f \leqslant 0 \\ \Phi(f / \sqrt{2}), & f \geqslant 0 \end{cases} \qquad (4.4)$$

where Φ is the error integral. The solution (4.4) enables us to obtain an acceptable agreement with the solutions for Z_0 and Θ_0 obtained by numerical methods.

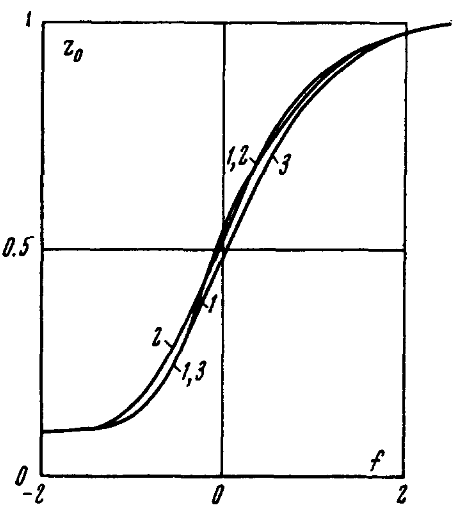

Fig. 2

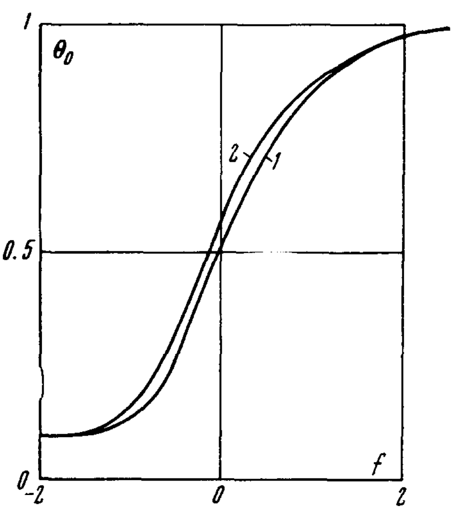

Fig. 3

Figure 2 depicts a plot of the function Z_0 for $\theta_w = 0.1$, the approximate solution according to the formula (4.4) denoted by *1* and the numerical solutions [6] for $\Lambda = 1$ and $\Lambda = 0.1$ denoted by *2* and *3*, respectively. Figure 3 gives the plots, also for $\theta_w = 0.1$, of Θ_0 for the approximate (formula (4.4)) and numerical [6] solutions denoted respectively by *1* and *2* (curve *2* corresponds to both, $\Lambda = 1$ and $\Lambda = 0.1$).

Comparison of these plots shows the divergence between them is small, we can therefore expect that the formulas (3.6) and (3.7) also yield an acceptable accuracy. We note that (4.4) and (3.6) do not contain Λ. This is due to the fact that the difference $(Z_0 - \vartheta_0)$ (or $[Z - (\rho_e / \rho)]$) respectively) is small. We also note that by virtue of the specific character of the method under consideration, the formula for θ (an analog of (3.7)) does not take the difference $(c_{p1} - c_{p2})$ into account. This can be avoided by either rejecting the complementary assumption that $\theta_{(0)} = 1/2\ (1 + \theta_w)$, or by using the function (3.7) instead of the discontinuous solution (2.2) to define c_f' in the equation for θ.

In conclusion we discuss the method of obtaining $(\theta_\eta')_w$ when $\xi = 0$, with the solution (4.4) taken into account. We note that

$$(\theta_\eta')_w = (\sqrt{z}\,\theta_f')_w = \frac{1}{\alpha}(1 - \theta_w) \Big/ \int_{-\infty}^{1} \exp\left(\alpha^2 \int_h^1 \frac{h'dh'}{\sqrt{z(h')}}\right) \frac{dh}{\sqrt{z}} \tag{4.5}$$

We decompose the integral appearing in the right-hand side into two terms

$$\int_{-\infty}^{1} \exp\left(\alpha^2 \int_h^1 \frac{h'dh'}{\sqrt{z(h')}}\right) \frac{dh}{\sqrt{z}} = \frac{1}{\alpha} \int_{-\alpha}^{0} \exp \varkappa(-\alpha, f) \frac{df}{\sqrt{z}} +$$

$$+ \frac{1}{\alpha} \exp \varkappa(-\alpha, 0) \int_0^\infty \exp \varkappa(0, f) \frac{df}{\sqrt{z}}$$

$$\varkappa(a, b) = -\int_a^b \frac{f df}{\sqrt{z}}, \qquad \alpha^2 \int_h^1 \frac{h'dh'}{\sqrt{z(h')}} = -\int_{-\alpha}^{f} \frac{f'df'}{\sqrt{z}} = \varkappa(-\alpha, f)$$

As in [6], we construct the combined solution in the following manner:

$$z = z_0 + Z_0 - \theta_w + \alpha^{-2} z_1 + O(\alpha^{-4} + \alpha^{-2\Lambda})$$

and respectively

$$-\varkappa(-\alpha, f) = -\alpha^2 \int_1^{-f/\alpha} (z_0 + \alpha^{-2} z_1)^{-1/2}\, h dh - \vartheta(f) + O(\alpha^{-4} + \alpha^{-2\Lambda}) \tag{4.6}$$

$$\vartheta(f) = -\int_{-\infty}^{f} (Z_0^{-1/2} - \theta_w^{-1/2}) f' df'$$

When $\alpha \to \infty$, the first term in (4.6) is written as follows:

$$\alpha^2 \int_1^{-f/\alpha} (z_0 + \alpha^{-2} z_1)^{-1/2}\, h dh = a^2 M - \frac{1}{2} f^2 \theta_w^{-1/2} + O(1/\alpha)$$

$$M = \int_0^1 (z_0 + \alpha^{-2} z_1)^{-1/2}\, h dh$$

Then in (4.5) we have

$$(\theta_\eta')_w = (1 - \theta_w) \exp(-\alpha^2 M) \Big/ \left\{ \int_{-\alpha}^{0} \exp\left[-\frac{1}{2} f^2 \theta_w^{-1/2} + \vartheta(f)\right] \frac{df}{\sqrt{z}} \left[1 + O(\alpha^{-1} + \alpha^{-2\Lambda})\right] + \right.$$

$$\left. + [\exp \vartheta(0)] \int_0^\infty \exp \varkappa(0, f) \frac{df}{\sqrt{z}} \left[1 + O(\alpha^{-2\Lambda} + \alpha^{-4})\right] \right\}$$

When $f = O(1)$, we have on the interval $f \leqslant 0$ $z_0 = \theta_w + O(\alpha^{-2\Lambda})$, therefore $z = Z_0 + O(\alpha^{-2} + \alpha^{-2\Lambda})$. Consequently

$$\int_{-\alpha}^{0} \exp\left[-\frac{1}{2} f^2 \theta_w^{-1/2} + \vartheta(f)\right] \frac{df}{\sqrt{z}} = \int_{-\infty}^{0} \exp\left[-\frac{1}{2} f^2 \theta_w^{-1/2} - \vartheta(f)\right] \frac{df}{\sqrt{Z_0}} \times$$
$$\times \left[1 + O(\alpha^{-2} + \alpha^{-2\Lambda})\right]$$

since the integrand function makes a significant contribution only in the region $f \leqslant 0$ and $f = O(1)$. Similarly we have

$$\int_0^\infty \exp \varkappa(0, f) \frac{df}{\sqrt{z}} = \int_0^\infty \exp \varkappa^*(0, f) \frac{df}{\sqrt{Z_0}} \left[1 + O(\alpha^{-2\Lambda})\right], \quad \varkappa^*(0, f) = -\int_{-\alpha}^{f} \frac{f' df'}{\sqrt{Z_0}}$$

Therefore

$$(\theta_\eta')_w = C_5^{-1}(1 - \theta_w) \exp(-\alpha^2 M)\left[1 + O(\alpha^{-1} + \alpha^{-2\Lambda})\right]$$

$$C_5 = \int_{-\infty}^{0} \exp\left[-\frac{1}{2} f^2 \theta_w^{-1/2} + \vartheta(f)\right] \frac{df}{\sqrt{Z_0}} + [\exp \vartheta(0)] \int_0^\infty \exp \varkappa^*(0, f) \frac{df}{\sqrt{Z_0}}$$

The constant C_5 given in the above formula should be calculated by numerical methods. Following [1] we can show that

$$M = \theta_w^{-1/2}(2\Lambda)^{-1} B\left(1/2, \frac{1}{\Lambda}\right) + \alpha^{-2}\Lambda^* + O(\alpha^{-4})$$

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